

Choquet integral in decision analysis – lessons from the axiomatization.

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Abstract

In [Timonin, 2016] we developed a general axiomatic treatment of a popular multicriteria decision model - the Choquet integral. This paper contains extensions of our results to the particular interesting special cases of the Choquet integral, analysis of some aspects of the Choquet integral model learning, and a discussion of the applications of our results in decision theory.

1 Introduction

In [Timonin, 2016] we developed a general axiomatic treatment of a popular multicriteria decision model - the Choquet integral. This paper contains extensions of our results to the particular interesting special cases of the Choquet integral, analysis of some aspects of the Choquet integral model learning, and a discussion of the applications of our results in decision theory. The Choquet integral is a powerful aggregation operator which lists many well-known models as its special cases. In this paper we look at these special cases and provide their axiomatic analysis. In cases where an axiomatization has been previously given in the literature, we connect the existing results with the framework that we have developed.

Next we turn to the question of learning, which is especially important for the practical applications of the model. So far, learning of the Choquet integral has been mostly confined to the learning of the capacity. Such an approach requires making a powerful assumption that all dimensions (e.g. criteria) are evaluated on the same scale, which is rarely justified in practice. Too often categorical data is given arbitrary numerical labels (e.g. AHP), and numerical data is considered cardinally and ordinally commensurate, sometimes after a simple normalization. Such approaches clearly lack scientific rigour, and yet they are commonly seen in all kinds of applications. We discuss the pros and cons of making such an assumption and look at the consequences which our uniqueness results have for the learning problems.

Finally, we revisit some of the applications we discussed in the Introduction. Apart from MCDA, which is the main area of interest for our results, we also discuss how the model can be interpreted in the social choice context. We look in detail at the state-dependent utility, and show how comonotonicity, central to the previous axiomatizations, actually implies state-independency in the Choquet integral model. We also discuss the

conditions required to have a meaningful state-dependent utility representation and show the novelty of our results compared to the previous methods of building state-dependent models.

2 Extensions

2.1 Ordinal models

Notable ordinal special cases of the Choquet integral are:

- Min/Max
- Order statistic (k -smallest element) OS_k
- Lattice polynomial p^{AB} .

Moreover, Min/Max are special cases of OS_k ($k = 1$ and $k = n$ correspondingly), and OS_k is a special case of the lattice polynomial model, as becomes evident from the following definitions.

Definition 1. \succsim can be represented by MIN, if exist value functions $\phi_i : X_i \rightarrow \mathbb{R}$ such that for all $x, y \in X$ we have

$$x \succsim y \iff \bigwedge_{i \in N} \phi_i(x_i) \geq \bigwedge_{i \in N} \phi_i(y_i), \quad (1)$$

where \bigwedge means minimum.

Definition 2. \succsim can be represented by MAX, if exist value functions $\phi_i : X_i \rightarrow \mathbb{R}$ such that for all $x, y \in X$ we have

$$x \succsim y \iff \bigvee_{i \in N} \phi_i(x_i) \geq \bigvee_{i \in N} \phi_i(y_i), \quad (2)$$

where \bigvee means maximum.

Definition 3. \succsim can be represented by an order statistic OS_k , if exist value functions $\phi_i : X_i \rightarrow \mathbb{R}$ such that for all $x, y \in X$ we have

$$x \succsim y \iff \phi_{(k)}(x_{(k)}) \geq \phi_{(k)}(y_{(k)}), \quad (3)$$

where $\phi_{(k)}(z_{(k)})$ stands for k th smallest element of $(\phi_1(z_1), \dots, \phi_n(z_n))$.

An order statistic can be written in a CNF and DNF-like¹ forms (e.g. Ovchinnikov, 1996):

$$OS_k = \bigwedge_{\substack{K \subset N \\ |K|=k}} \bigvee_{i \in K} \phi_i(x_i) = \bigvee_{\substack{K \subset N \\ |K|=n-k+1}} \bigwedge_{i \in K} \phi_i(x_i). \quad (4)$$

Obviously, MIN and MAX are particular cases of OS_k with $k = 1$ and $k = n$ correspondingly.

¹Conjunctive normal form and disjunctive normal form.

Definition 4. \succsim can be represented by a lattice polynomial p^{AB} , if exist value functions $\phi_i : X_i \rightarrow \mathbb{R}$ such that for all $x, y \in X$ we have

$$x \succsim y \iff p^{AB}(\phi_1(x_1), \dots, \phi_n(x_n)) \geq p^{AB}(\phi_1(y_1), \dots, \phi_n(y_n)), \quad (5)$$

where $p^{AB}(\phi_1(z_1), \dots, \phi_n(z_n))$ is an expression which includes elements of $(\phi_1(z_1), \dots, \phi_n(z_n))$ and symbols \vee and \wedge .

We can write any lattice polynomial in DNF and CNF as well:

$$p^{AB}(\phi_1(z_1), \dots, \phi_n(z_n)) = \bigwedge_{K \subset \mathcal{A}} \bigvee_{i \in K} \phi_i(x_i) = \bigvee_{M \subset \mathcal{B}} \bigwedge_{i \in M} \phi_i(x_i), \quad (6)$$

where $\mathcal{A} \subset 2^N$ and $\mathcal{B} \subset 2^N$ are some collection of subsets of N . Obviously, order statistic, hence MIN and MAX are special cases of an order polynomial.

The following result states that all aforementioned models are special cases of the Choquet integral.

Theorem 1 (Murofushi and Sugeno, 1993). *The Choquet integral with respect to a capacity ν is a lattice polynomial function if and only if ν is a 0–1 capacity (i.e. only takes values 0 or 1). Moreover, any lattice polynomial function on \mathbb{R} is a Choquet integral with respect to a 0–1 capacity.*

2.2 Previous characterizations of the ordinal models

Some known characterizations of the models presented in the previous section are due to Bouyssou et al. [2002], see also Sounderpandian [1991] and Segal and Sobel [2002].

Theorem 2 (Bouyssou et al., 2002). *\succsim can be represented by MAX if \succsim is a weak order and the following equivalent conditions hold:*

1. For all $i \in N, x_i, y_i \in X_i, a_{-i}, b_{-i} \in X_{-i}$ and $w \in X$, we have

$$[x_i a_{-i} \succsim w] \Rightarrow [y_i a_{-i} \succsim w \text{ OR } x_i b_{-i} \succsim w] \quad (7)$$

2. For all $x, y \in X, i \in N$:

$$[x_i y_{-i} \succsim x] \text{ OR } [y_i x_{-i} \succsim x] \quad (8)$$

3. For all $i \in N, y_i \in X_i, z_{-i} \in X_{-i}, x \in X$:

$$[y_i x_{-i} \succ x] \Rightarrow [y_i z_{-i} \succ x]. \quad (9)$$

Theorem 3 (Bouyssou et al., 2002). *\succsim can be represented by MIN if \succsim is a weak order and the following equivalent conditions hold:*

1. For all $i \in N, x_i, y_i \in X_i, a_{-i}, b_{-i} \in X_{-i}$ and $w \in X$, we have

$$[w \succ x_i a_{-i}] \Rightarrow [w \succ y_i a_{-i} \text{ OR } w \succ x_i b_{-i}] \quad (10)$$

2. For all $x, y \in X, i \in N$:

$$[x \succ x_i y_{-i}] \text{ OR } [x \succ y_i x_{-i}] \quad (11)$$

3. For all $i \in N, y_i \in X_i, z_{-i} \in X_{-i}, x \in X$:

$$[x \succ y_i x_{-i}] \Rightarrow [x \succ y_i z_{-i}]. \quad (12)$$

Theorem 4 (Bouyssou et al., 2002). \succ can be represented by OS_{n-1} if \succ is a weak order and the following equivalent conditions hold:

1. For all $i, j \in N (i \neq j), x_i, y_i \in X_i, x_j, y_j \in X_j, a_{-i} \in X_{-i}, b_{-j} \in X_{-j}, c_{-ij} \in X_{-ij}$ and $w \in X$, we have

$$[x_i a_{-i} \succ w \text{ AND } x_j b_{-j} \succ w] \Rightarrow [y_i a_{-i} \succ w \text{ OR } y_j b_{-j} \succ w \text{ OR } x_{ij} c_{-ij} \succ w] \quad (13)$$

2. For all $x, y \in X, i, j \in N (i \neq j)$:

$$[x_i y_{-i} \succ x \text{ AND } x_j y_{-j} \succ x] \text{ OR } [y_{ij} x_{-ij} \succ x] \quad (14)$$

3. For all $x, y \in X$, all $i, j \in N (i \neq j)$, and all $z_{-ij} \in X_{-ij}$:

$$[y_i x_{-i} \succ x \text{ AND } y_j x_j \succ x] \Rightarrow [y_{ij} z_{-ij} \succ x]. \quad (15)$$

2.3 Unified characterization of the ordinal models: p^{AB} and sub-cases

Since MIN and MAX are special cases of OS_k , which in turn is a special case of the lattice polynomial models p^{AB} , it is desirable to build a unified characterization for all of them. In this section we provide some steps towards such result.

Theorem 5. \succ can be represented by a lattice polynomial p^{AB} if \succ is a weak order, satisfies **A2**, and for any $w, x \in X$ exist $K \in \mathcal{A}, M \in \mathcal{B}$ with $K \cap M \neq \emptyset$, such that for any $a_{-K} \in X_{-K}$ and $b_{-M} \in X_{-M}$ we have:

$$\begin{cases} w \succ x \Rightarrow w \succ a_{-K} x_K, K \in \mathcal{A}, \\ x \succ w \Rightarrow b_{-M} x_M \succ w, M \in \mathcal{B}. \end{cases} \quad (16)$$

Note that, because sets \mathcal{A} and \mathcal{B} are finite, the axiom can also be re-written similar to the conditions in the previous section, i.e. using “OR” statements. However, we feel this form is more compact. Particular cases of the above axiom include OS_k and MIN/MAX.

Lemma 1. \succ can be represented by OS_k if \succ is a weak order, satisfies **A2**, and for any $w, x \in X$ there exist $K : K \subset N, |K| = k$ and $M : M \subset N, |M| = n - k + 1$ with $K \cap M \neq \emptyset$, such that for any $a_{-K} \in X_{-K}$ and $b_{-M} \in X_{-M}$ we have

$$\begin{cases} w \succ x \Rightarrow w \succ a_{-K} x_K, \\ x \succ w \Rightarrow b_{-M} x_M \succ w. \end{cases} \quad (17)$$

Lemma 2. \succsim can be represented by MIN if \succsim is a weak order and for any $w, x \in X$ exists $i \in N$, such that for any $a_{-i} \in X_{-i}$ we have

$$\begin{cases} w \succsim x \Rightarrow w \succsim a_{-i}x_i, \\ x \succsim w \Rightarrow x \succsim w. \end{cases} \quad (18)$$

Lemma 3. \succsim can be represented by MAX if \succsim is a weak order and for any $w, x \in X$ exists $i \in N$, such that for any $b_{-i} \in X_{-i}$ we have

$$\begin{cases} w \succsim x \Rightarrow w \succsim x, \\ x \succsim w \Rightarrow b_{-i}x_i \succsim w. \end{cases} \quad (19)$$

The second condition in two last lemmas is trivial and is given only to emphasize the similarity of the axiom to the one used above. Note also, that the first conditions in MIN/MAX characterizations are identical to those given in Section 2.2.

Although the condition in two last lemmas is sufficient for characterization of MIN and MAX, in general, variations of (16) are not powerful enough to characterize p^{AB} and OS_k . One reason for this is that in the MIN/MAX case the axioms imply our **A2** (the axiom that is called **AC1** in Bouyssou et al. [2009]) – in other words they imply existence of weak orders on individual dimensions. This does not seem to be the case for the p^{AB} and OS_k conditions that we gave. Hence, we had to add **A2** to the first two results.

2.4 Characterization of the ordinal models in our framework

In [Timonin, 2016] we gave details of the construction of the Choquet integral for cases when every subset X^{S_i} has only one essential variable. We now provide more details on this result.

Lemma 4. Let the conditions of Theorem ?? hold and let there be only one essential variable on each X^{S_a} . Then, ν is a 0–1 capacity.

Proof. This immediately follows by construction (see Section ??). As at every $x \in X$ we have $C(\nu, x) = f_i(x_i)$, where i is the variable essential on $X^{S_i} \ni x$, by the definition of the Choquet integral and monotonicity of ν it follows that ν only takes values 0 and 1. \square

Lemma 5. Let the conditions of Theorem ?? hold and let there be only one essential variable on each X^{S_a} .

- \succsim can be represented by p^{AB} ;
- If the essential variable on every X^{S_i} is the **R**-minimal one, then \succsim can be represented by MIN;
- If the essential variable on every X^{S_i} is the **R**-maximal one, then \succsim can be represented by MAX;
- If the essential variable on every X^{S_i} is the **R**- k -minimal one, then \succsim can be represented by $OS - k$.

Proof. The first statement follows from Theorem 1. Other follow by construction and from the uniqueness properties of the representation (??) in the ordinal case (see Theorem ??). If \mathbf{S} ordering is incomplete, then only one \mathbf{R} ordering can exist which does not contradict **A3, A7** and the condition that only one variable is essential on every X^{S_a} . This follows from the uniqueness of the capacity and the uniqueness properties of the value functions. \square

2.5 Cardinal models

The particular cases of the Choquet integral in the case of cardinal value functions are related to the convexity of the capacity. We give a characterization of the convex capacity (the concave case is easily obtainable by reversing the preference). Note that in the two-dimensional case, the class of the Choquet integrals with respect to convex capacities coincides with the class of Gilboa–Schmeidler maximin models. In the general case of n dimensions, every Choquet integral with respect to a convex capacity is a Gilboa–Schmeidler model – the integral is a minimum of integrals with respect to probability distributions from the capacity’s core [Gilboa and Schmeidler, 1994] – but not other way round.

To our knowledge, this is the first result which characterizes convexity of a capacity using only the primitives of \succsim and works in ordinal or mixed as well as purely cardinal cases, i.e. it is suitable for situations when standard sequences are not available.

Theorem 6. *Let conditions **A1–A9** and structural assumptions hold. Then, we have*

A10 – Convexity *For all $i, j \in N$ and for all $a_i, b_i, c_i, d_i \in X_i$, $p_j, q_j, r_j, s_j \in X_j$, and all $z_{-ij} \in X_{-ij}$ we have*

$$\left. \begin{array}{l} a_i p_j z_{-ij} \sim b_i q_j z_{-ij} \\ a_i r_j z_{-ij} \sim b_i s_j z_{-ij} \\ c_i p_j z_{-ij} \sim d_i q_j z_{-ij} \\ d_i \succsim_i c_i \\ r_j \succsim_j s_j \end{array} \right\} \Rightarrow c_i r_j z_{-ij} \succsim d_i s_j z_{-ij}, \quad (20)$$

provided $j \mathbf{R} i$ at $a_i p_j z_{-ij}, b_i q_j z_{-ij}, a_i r_j z_{-ij}, \sim b_i s_j z_{-ij}, c_i p_j z_{-ij}, d_i q_j z_{-ij}$ and $i \mathbf{R} j$ at $c_i r_j z_{-ij}$ and $d_i s_j z_{-ij}$,

if and only if ν is a convex capacity.

Proof. Since conditions **A1–A9** and structural assumptions hold, there exists a Choquet integral representation of \succsim . We can use it to prove the statement of the theorem. A capacity is convex if for all $i, j \in N, A \subset N, i \neq j$ we have [Chateauneuf and Jaffray, 1989]:

$$\sum_{i,j \in B \subset A} m(B) \geq 0. \quad (21)$$

First, let $c_i r_j z_{-ij} \prec d_i s_j z_{-ij}$. We can write the conditions above using the Möbius form of the Choquet integral. All subsets of N can be separated into four groups:

- $A : A \ni i, A \not\ni j$

- $A : A \ni j, A \not\ni i$
- $A : A \ni i, A \ni j$
- $A : A \not\ni i, A \not\ni j$.

Hence, the value function for each of the points in the axiom can be written as follows. For example, for $a_i p_j z_{-ij}$ (note that we have merged $A : A \ni i, A \not\ni j$ and $A : A \ni i, A \ni j$ groups by virtue of $j \mathbf{R} i$ at $a_i p_j z_{-ij}$):

$$\sum_{A \ni i} m(A) \min_{k \in A - ij} [f_i(a_i), f_k(z_k)] + \sum_{\substack{A \ni j \\ A \not\ni i}} m(A) \min_{k \in A - ij} [f_i(p_j), f_k(z_k)] + \sum_{A \ni i, j} m(A) \min_{k \in A - ij} [f_k(z_k)]. \quad (22)$$

Writing down all four conditions like this and after some trivial algebraic transformations which we omit in the name of readability (sum first two conditions, add to the sum of the last two conditions and simplify), we get

$$\begin{aligned} & \sum_{A \ni i, j} m(A) \left(\min_{k \in A - ij} [f_i(d_i), f_k(z_k)] - \min_{k \in A - ij} [f_i(c_i), f_k(z_k)] \right) \\ & + \sum_{A \ni i, j} m(A) \left(\min_{k \in A - ij} [f_j(r_j), f_k(z_k)] - \min_{k \in A - ij} [f_j(s_j), f_k(z_k)] \right) < 0. \end{aligned} \quad (23)$$

We will show that both summands of the above expression are non-negative. Consider

$$\sum_{A \ni i, j} m(A) \left(\min_{k \in A - ij} [f_i(d_i), f_k(z_k)] - \min_{k \in A - ij} [f_i(c_i), f_k(z_k)] \right). \quad (24)$$

The difference $f_i(d_i), f_k(z_k)] - \min_{k \in A - ij} [f_i(c_i), f_k(z_k)]$ is

- always non-negative, as $d_i \succ_i c_i$
- maximal, when $A = \{i, j\}$
- non-increasing as A grows larger.

Note that, by convexity, $m(\{i, j\}) \geq 0$. Hence,

$$m(\{i, j\}) \left(\min_{k \in \emptyset} [f_i(d_i), f_k(z_k)] - \min_{k \in \emptyset} [f_i(c_i), f_k(z_k)] \right) = m(\{i, j\}) (f_i(d_i) - f_i(c_i)) \geq 0. \quad (25)$$

Next, find a maximal $f_{k^1}(z_{k^1}), k^1 \in N \setminus i, j$. Note that in the above expression we will only have one element $\min_{k \in A - ij} [f_i(d_i), f_k(z_k)] - \min_{k \in A - ij} [f_i(c_i), f_k(z_k)]$ where k^1 is not redundant (since it's maximal). We get

$$\begin{aligned} & m(\{i, j\}) (f_i(d_i) - f_i(c_i)) + m(\{i, j, k^1\}) (\min[f_i(d_i), f_{k^1}(z_{k^1})] - \min[f_i(c_i), f_{k^1}(z_{k^1})]) \\ & \geq [m(\{i, j\}) + m(\{i, j, k^1\})] (\min[f_i(d_i), f_{k^1}(z_{k^1})] - \min[f_i(c_i), f_{k^1}(z_{k^1})]) \geq 0. \end{aligned} \quad (26)$$

The first inequality is since $m(\{i, j\}) \geq 0$ and the second is since $m(\{i, j\}) + m(\{i, j, k^1\}) \geq 0$, by convexity criterion. Now pick the second largest $f_{k^2}(z_{k^2}), k^2 \in N \setminus i, j, k^1$. Using the same arguments we get

$$\begin{aligned}
& m(\{i, j\}) (f_i(d_i) - f_i(c_i)) + m(\{i, j, k^1\}) (\min[f_i(d_i), f_{k^1}(z_{k^1})] - \min[f_i(c_i), f_{k^1}(z_{k^1})]) \\
& + m(\{i, j, k^2\}) (\min[f_i(d_i), f_{k^2}(z_{k^2})] - \min[f_i(c_i), f_{k^2}(z_{k^2})]) \\
& + m(\{i, j, k^1, k^2\}) (\min[f_i(d_i), f_{k^2}(z_{k^2})] - \min[f_i(c_i), f_{k^2}(z_{k^2})]) \geq [m(\{i, j\}) + m(\{i, j, k^1\}) \\
& + m(\{i, j, k^2\}) + m(\{i, j, k^1, k^2\})] (\min[f_i(d_i), f_{k^2}(z_{k^2})] - \min[f_i(c_i), f_{k^2}(z_{k^2})]) \\
& \geq 0.
\end{aligned} \tag{27}$$

Continuing like this we can add more and more elements and eventually conclude that

$$\sum_{A \ni i, j} m(A) \left(\min_{k \in A - ij} [f_i(d_i), f_k(z_k)] - \min_{k \in A - ij} [f_i(c_i), f_k(z_k)] \right) \geq 0. \tag{28}$$

Similarly,

$$\sum_{A \ni i, j} m(A) \left(\min_{k \in A - ij} [f_j(r_j), f_k(z_k)] - \min_{k \in A - ij} [f_j(s_j), f_k(z_k)] \right) \geq 0. \tag{29}$$

Hence we have shown that the axiom necessarily holds if the capacity is convex. To show the inverse, assume that the axiom holds on X . Writing down conditions of the axiom and simplifying as before, we get that everywhere on X we should have

$$\begin{aligned}
& \sum_{A \ni i, j} m(A) \left(\min_{k \in A - ij} [f_i(d_i), f_k(z_k)] + \min_{k \in A - ij} [f_j(r_j), f_k(z_k)] \right) \\
& \geq \sum_{A \ni i, j} m(A) \left(\min_{k \in A - ij} [f_i(c_i), f_k(z_k)] - \min_{k \in A - ij} [f_j(s_j), f_k(z_k)] \right).
\end{aligned} \tag{30}$$

Assume i, j interact. If this is not the case, the convexity criterion is trivially satisfied for i, j as all $m(A)$ in the expression above are 0 (see Lemma ??). Assume also all variables are in the same interaction group. If this is not the case, $m(A)$ for A containing variables not in the same interaction group as i, j are again 0, and can be discarded.

With this assumption made, we can now pick points, such that $f_i(\cdot)$ and $f_j(\cdot)$ are the smallest value functions. Hence, the above expression reduces to

$$[f_i(d_i) + f_j(r_j)] \sum_{A \ni i, j} m(A) \geq [f_i(c_i) + f_j(s_j)] \sum_{A \ni i, j} m(A). \tag{31}$$

Since $[f_i(d_i) + f_j(r_j)] \geq [f_i(c_i) + f_j(s_j)]$, we conclude that $\sum_{i, j \in A \subset N} m(A) \geq 0$.

Now pick points such that only $f_{k^1}(z_{k^1})$ is less than $f_i(\cdot)$ and $f_j(\cdot)$. We get

$$\begin{aligned}
& [f_i(d_i) + f_j(r_j)] \sum_{\substack{A \ni i, j \\ A \not\ni k^1}} m(A) + 2f_{k^1}(z_{k^1}) \sum_{A \ni i, j, k^1} m(A) \\
& \geq [f_i(c_i) + f_j(s_j)] \sum_{\substack{A \ni i, j \\ A \not\ni k^1}} m(A) + 2f_{k^1}(z_{k^1}) \sum_{A \ni i, j, k^1} m(A),
\end{aligned} \tag{32}$$

or

$$[f_i(d_i) + f_j(r_j)] \sum_{\substack{A \ni i,j \\ A \not\ni k^1}} m(A) \geq [f_i(c_i) + f_j(s_j)] \sum_{\substack{A \ni i,j \\ A \not\ni k^1}} m(A). \quad (33)$$

From this we conclude that $\sum_{i,j \in A \subset N \setminus k^1} m(A) \geq 0$.

Continuing like this we can check all necessary sums for the convexity condition and for all pairs i, j . So, we have shown that the capacity is convex provided the axiom holds. \square

3 Learning the Choquet integral

Learning the model means deriving model parameters from data. This step is essential in any practical application, and it is normally performed towards at least one of two goals: analysis of the data, by means of interpreting model parameters, or prediction – in other words, “training” the model on some dataset to use it with some other data.

It is well known that the quality of fit of a model depends on the model complexity and the available data. Learning a very complex model using only a few data points would not achieve satisfactory results, just as using a very simple model might conceal some important properties of a large and complicated dataset.

An important aspect of the learning process is its computational viability. Indeed, from the practical perspective, using a simpler but faster model which is capable of delivering approximate answers in real-time fashion, might be preferable to employing a more precise but also more expensive model which takes hours or days to be built.

In this section we look at various aspects of the Choquet integral learning and emphasize the consequences which our axiomatization results have for this process. We start by an overview of the current learning techniques and then look at difficulties which arise when learning the Choquet integral model in the full generality.

To learn the Choquet integral we need to derive two parts of the model from data:

- value functions $f_i : X_i \rightarrow R$, and
- capacity ν .

The following sections provide details on each of these components.

3.1 Learning the capacity

The majority of the theoretical and applied literature so far has concentrated on learning (“identification”) of the capacity *only*. In this approach, the value functions are assumed as given. Normally, for numerical coordinates $f_i(x_i) = x_i$ are taken (probably after some rescaling). For categorical data, sometimes arbitrary numerical labels are used (see e.g. AHP), although the theoretical problems of this approach are quite apparent.

A good review of the existing methods of capacity construction can be found in Grabisch et al. [2008]. In the majority of cases, the learning process is based on minimization of some loss function (MSE, MAE, or similar), or on finding the extremum of some meaningful expression, such as variance or entropy.

Typically, data is used to formulate constraints on the space of possible parameters (i.e. capacities). For example, if $x \succ y$, then ν must be such that $C(\nu, f(x)) \geq C(\nu, f(y))$ (remember the value functions are considered known). Since the integral is a linear function of the capacity, we get linear constraints. Eventually the polyhedron of all possible capacities is defined by the following data:

Learning set. Pairwise preferences between elements of the “learning set” X .

Criteria importance. The most intuitive way to describe a multicriteria model qualitatively is, perhaps, to define the relative weights of its components. The process is semantically similar to that for additive models; however, due to non-additivity we can not rely only on values for singletons any more, but must also take into account all other subsets of N .

Criteria interaction. A more complicated type of knowledge about criteria is the character of their combined influence. In particular, criteria can complement each other, which is also known under the name of positive synergy, or else be redundant (resp. negative synergy).

Veto and favour criteria. Sometimes the model also includes criteria of an immense importance, so that the alternatives having low valuations on them will also inevitably receive low overall judgements. This kind of criterion is usually called “veto” in the literature. The opposite situation is having a criterion (or criteria) such that a high value on them automatically justifies a high overall valuation. Such elements are called “favour”.

Complexity controls. Often it is deemed that interactions in groups larger than k can be ignored to improve the computational properties of the model. The mechanism which allows us to achieve this is called k -additivity. Most frequently, 2-additive capacities are used.

The following indices were originally applied for behavioral analysis of non-additive measures. However, they also allow us to formulate and solve the inverse problem of capacity identification (see Marichal and Roubens, 2000 and references therein).

Definition 5 (Shapley, 1953). *The Shapley value is an additive measure $\phi_\nu : 2^N \rightarrow [0, 1]$ defined as*

$$\phi_\nu(i) = \sum_{T \subset N \setminus i} \frac{(|N| - |T| - 1)! |T|!}{|N|!} [\nu(T \cup i) - \nu(T)]. \quad (34)$$

It can also be expressed via the Möbius transform coefficients:

$$\phi_m(i) = \sum_{T \subset N \setminus i} \frac{1}{|T| + 1} m(T \cup i). \quad (35)$$

The semantic interpretation given to the Shapley value of a criterion $i \in N$ in the literature is the relative importance of the said criterion in the decision problem. More formally, it amounts to the average marginal input of that criterion to all subsets of N . Being a probability measure, the Shapley value sums up to 1 over all $i \in N$. Table 1

Table 1: Criteria importance modelling

The criterion i is more important than j	$\phi_\nu(i) - \phi_\nu(j) \geq \delta_{SH}$
Criteria i and j are equally important	$-\delta_{SH} \geq \phi_\nu(i) - \phi_\nu(j) \leq \delta_{SH}$

demonstrates how the Shapley value can be used in capacity identification problems (δ_{SH} is some small value – the indifference coefficient). Intuition about the relative importance of a criterion can be expressed as $\phi_\nu(i) = k$ or $\phi_\nu(i) \in [k^l, k^u]$, although, just like in the additive case, doing so is not strictly sensible.

The measure of criteria interaction character and strength was introduced by Murofushi and Soneda [1993] for pairs of elements and later generalized by Grabisch [1997a].

Definition 6. *The interaction index of a subset $T \subset N$ is defined as*

$$I_\nu(T) = \sum_{k=0}^{|N|-|T|} \xi_k^{|T|} \sum_{K \subset Z \setminus T, |K|=k} \sum_{L \subset T} (-1)^{|T|-|L|} \nu(L \cup K), \quad (36)$$

where

$$\xi_k^p = \frac{(|N|-k-p)! k!}{(|N|-p+1)!}. \quad (37)$$

For practical problems we are particularly interested in the index expression for pairs $\{i, j\}$:

$$I_\nu(ij) = \sum_{T \subset N \setminus ij} \xi_{|T|}^2 [\nu(T \cup ij) - \nu(T \cup i) - \nu(T \cup j) + \nu(T)], \quad (38)$$

or, when expressed with the Möbius transform coefficients:

$$I_m(ij) = \sum_{T \subset N \setminus ij} \frac{1}{|T|+1} m(T \cup ij). \quad (39)$$

The interaction index for singletons coincides with the Shapley value. The index can be interpreted as the degree of interaction between elements in the set T . Its values lie in the interval $[-1; 1]$, with 1 corresponding to the maximal positive interaction (complementarity), and -1 , accordingly, to the maximal negative interaction (redundancy). Table 2 summarizes index usage in identification problems.

Table 2: Modelling criteria interactions

Criteria i and j complement each other	$0 \leq I_\nu(i, j) \leq 1$
Criteria i and j complement each other stronger than k and l	$I_\nu(i, j) - I_\nu(k, l) \geq \delta_I$
Criteria i and j interact in a way similar to k and l	$-\delta_I \geq I_\nu(i, j) - I_\nu(k, l) \leq \delta_I$

To model “veto” and “favour” criteria we can proceed in the following way [Grabisch, 1997b]. If some criterion i is a “veto” one, then

$$\nu(A) = 0 \quad \forall A \not\supseteq i. \quad (40)$$

Else, if some criterion i is a “favour” one, then

$$\nu(A) = 1 \quad \forall A \supseteq i. \quad (41)$$

Finally, if the problem allows us to employ a learning set, the DM might be asked to express his preferences with regard to its elements. In an identification problem this corresponds to linear constraints (since the integral is linear in ν) outlined in Table 3.

Table 3: Preferences over learning set objects

The alternative z_1 is preferred to z_2	$C(\nu, f(z_1)) - C(\nu, f(z_2)) \geq \delta_{LS}$
The DM is indifferent between z_1 and z_2	$-\delta_{LS} \geq C(\nu, f(z_1)) - C(\nu, f(z_2)) \leq \delta_{LS}$

Having the available information expressed as a set of linear constraints we obtain the set \mathcal{U} . Summing up the results of the previous section, \mathcal{U} can be written down as shown in equation (42).

Notably, all constraints are linear, and thus the set \mathcal{U} is a polyhedron in $\mathbb{R}_+^{2^n}$. Its dimension can be reduced to $2^n - 2$ if we exclude the \emptyset and N coordinates, which have fixed values. It can be reduced even further by using k -additive capacities which, however, is not always possible. By solving the feasibility problem

$$\begin{aligned} & \min_{\nu} 1 \\ & \text{s.t. } \nu \in \mathcal{U}, \end{aligned} \quad (43)$$

we can check if there exists at least one capacity compliant with the given data. If such capacity cannot be found, the following problem can be solved:

$$\begin{aligned} & \min_{\nu} \mathcal{L}(\mathcal{U}) \\ & \text{s.t. } \nu \text{ is a capacity,} \end{aligned} \quad (44)$$

where $\mathcal{L}(\mathcal{U})$ is some loss function of the data (e.g. the number of preference reversals). The loss function, whether an error-based one or some other as mentioned above, is typically a convex function, so the optimization problem is quite efficient. If the model is built for forecasting purposes, regularization techniques can also be used [Tehrani and Huellermeier, 2013, Tehrani et al., 2012, 2011a,b]. Additionally, identification problems can have more than one solution, which induces the problem discussed below.

3.2 Learning the value functions

Learning the value functions on the other hand is a different matter. Let us consider first how the process is performed in the additive value function model. Recall that the model

\mathcal{U} :

Information from the DM

$$\phi_\nu(i) - \phi_\nu(j) \geq \delta_{SH}, \quad i, j \in 1, \dots, n$$

...

$$-\delta_{SH} \geq \phi_\nu(i) - \phi_\nu(j) \leq \delta_{SH}, \quad i, j \in 1, \dots, n$$

...

$$I_\nu(i, j) - I_\nu(k, l) \geq \delta_I, \quad i, j \in 1, \dots, n$$

...

$$-\delta_I \geq I_\nu(i, j) - I_\nu(k, l) \leq \delta_I, \quad i, j \in 1, \dots, n$$

...

$$C(\nu, f(z_i)) - C(\nu, f(z_j)) \geq \delta_{LS}, \quad i, j \in 1, \dots, n \quad (42)$$

...

$$-\delta_{LS} \geq C(\nu, f(z_i)) - C(\nu, f(z_j)) \leq \delta_{LS}, \quad i, j \in 1, \dots, n$$

...

$$\nu(A) = 1, \forall A \supset \text{favour criteria}$$

$$\nu(A) = 0, \forall A \not\supset \text{veto criteria}$$

Technical constraints

$$\nu(\emptyset) = 0$$

$$\nu(N) = 1$$

$$\nu(B) \geq \nu(A) \quad \forall B \subset A \subset N$$

Additional constraints

k – additivity. Not always applicable.

Figure 1: Encoding the information as constraints on the set of capacities

has the following form:

$$x \succ y \iff \sum_{i=1}^n f_i(x_i) \geq \sum_{i=1}^n f_i(y_i). \quad (45)$$

The data in such a learning problem is typically given as pairwise preferences for some points from the set X . The resulting problem is then an LP, because additive value functions are linear with respect to each f_i that we are aiming to learn. A well-known family of learning methods related to learning of the additive value models are called the “UTA methods” [Siskos et al., 2005]. The value functions are assumed to be linear interpolations of the learning points (i.e. they are piecewise linear), but sometimes polynomial or spline-based versions are used [Sobrie et al., 2016]. Still, the process remains computationally efficient.

Note that the value functions learned in this manner do not provide any “qualitative” information about the data to the analyst. They can be used for forecasting purposes,

but due to the restrictions of the additive model, no statements about the “importance” of criteria or similar notions can be made. In contrast, learning value functions and the capacity in the Choquet integral is valuable even if the value functions are learned in a non-parametric manner. Indeed, it is the capacity that is capable of showing the qualitative relations between criteria of the multidimensional problem, as is to some extent attested by the majority of the existing practical applications. However, this process has two complications: the computational complexity and the confounding of the capacity and the value functions.

As mentioned above, the vast majority of the theoretical and practical contributions to the literature assume the existence of value functions, or what is the same, of a common scale on which all attributes of the problem are being measured. This is clearly a very strong assumption, but it also leads to a significant simplification of the learning process. Indeed, in this case we only need to learn the capacity, which is generally a convex minimization problem. In contrast, when learning *both* the capacity and the value functions, we must solve a difficult non-convex optimization problem. Only a few papers have attempted to tackle this issue [Angilella et al., 2004, Goujon and Labreuche, 2013, Angilella et al., 2015], all of them offering some heuristic methods and small-scale examples. This is not surprising. Indeed, consider the data point $x \succsim y$ for some $x, y \in X$. In the Choquet integral model, it is represented by the following expression: $C(\nu, f(x)) \geq C(\nu, f(y))$. Since the integral is a sum of products of elements of ν and $f(x)$, the constraint is not linear in contrast to the case where only capacity is considered unknown. Moreover, it is generally non-convex. Hence, the process of construction of the capacity and the value functions involves solving a non-convex optimization problem, which is known to be computationally hard.

3.3 Confounding of the capacity and the value functions

The second issue in the Choquet integral learning problems is the non-uniqueness of the resulting capacity. Even in cases where only capacity is being learned, the exponential number of the coefficients ($2^n - 2$, excluding $\nu(\emptyset)$ and $\nu(N)$) means that the task of model learning quickly becomes very difficult as the number of dimensions of the model increases. Typically a learning dataset which is not sufficiently large does not allow the capacity to be learned in a precise way. This is a very well-known problem in general learning theory [Hüllermeier and Tehrani, 2012] and it can be addressed by a number of methods. Among these we can mention the general regularization approaches [Tehrani and Huellermeier, 2013, Tehrani et al., 2012, 2011a,b], but also some specialized methods which can be applied when the model is used in particular applications, such as sorting [Angilella et al., 2015, 2010]. Additionally, a number of methods were developed for robust decision making with the Choquet integral. Thus, in Timonin [2013] we proposed an algorithm for regret-minimizing optimization when the capacities are only known to belong to a certain set, whereas Benabbou et al. [2015, 2014] looked at the problem of the robust capacity construction using interactive data.

Axiomatization introduced in this work adds another level of complexity to the uniqueness problem. Indeed, the uniqueness results state that meaningful and unique decomposition of the capacity and the value functions is only possible when the model exhibits sufficient levels of non-separability. In particular, pairwise violation of ij -triple cancella-

tion should be present to a sufficient extent to obtain a unique capacity (in particular, all variables should be in the same interaction group, see Section ??). Thus, even an indefinite amount of data, not containing a sufficiently rich structure of preferences, would lead to a strongly non-unique capacity. In fact, it is easy to show that the capacity in such cases can be taken almost arbitrarily. Consider the extreme example, when there is no pairwise interaction in the model. In this case, we have n interaction groups of size 1 or, in other words, an additive value model. In the expression $w_1 f_1(x_1) + \dots + w_n f_n(x_n)$ we can arbitrarily change the “weights” w_i by compensating their increase or decrease by a proportional change in f_i . The whole model can then be rescaled so that the weights sum up to 1. It is trivial that these modifications do not affect the validity of the representations.

Non-uniqueness of the capacity is not necessarily a problem for prediction applications; however, qualitative conclusions, commonly made based on capacity indices, become meaningless. For example, consider the paper of Li et al. [2012]. Here, data from hotel evaluations on the tripadvisor website is analysed with the Choquet integral. Each hotel is reviewed based on several criteria, such as price, location, etc. In addition, every hotel gets an overall mark, which allows the authors to construct the relation between general attractiveness of the hotel and its particular features or their combinations. Reviewers are categorized into several social groups (“American businessmen”, “European families”, etc). The paper shows which attributes and combinations of attributes are important for every group by finding capacities that provide the best fit of the 2-additive Choquet integral to the corresponding dataset. Shapley values and interaction indices of these capacities provide the required information.

From our perspective, the important point is that the evaluations are *assumed* to be on the same scale. Every criterion is given from one to five stars, and so is the global evaluation. Of course it seems not completely unreasonable to suggest that various incommensurable notions such as “5 minutes from the train station” and “very clean” are somehow mapped onto a global “satisfaction” scale in the mind of the reviewer, indeed there are many examples of such “cross-modality” mappings in the psychological literature (see Section 4.2). However, there is no real evidence supporting this claim, and we can also assume that stars on each dimension signify just the ranking within the dimension itself and not across dimensions as the authors conjecture. The other consequence of such assumption is that the scale is equispaced, in the sense that the (cardinal) difference between one and two stars is the same as between two and three and between four and five. Apparently, this does not have to be true and often is not.

The possibility to fit not only the capacity but also the value functions resolves these methodological issues. Apparently, it should also improve the quality of the fit. However, in cases when we assume a common scale, the lack of interaction between certain criteria is not an issue – we still obtain a unique capacity (see also axiomatizations in Wakker, 1989 and Schmeidler, 1989) and corresponding indices, which would show a lack of interaction. In contrast, without the commensurability assumption, having even two interaction groups would mean that we are not able to talk about “criteria importance” globally, but only within these groups. The problem here is not with the tools used for capacity interpretation, in this case the Shapley value, but rather comes from the limitations of the model per se. Unfortunately, it is not easy to see how this problem can be resolved, as it is in fact the same issue as the impossibility of meaningfully using the notion of “criteria weights” in the additive model [Bouyssou et al., 2000] (Chapter 6). It

is notable, however, that the value of the interaction index would remain zero for any two elements from different interaction groups, no matter how we transform the capacity!²

4 Interpretations and discussion

Motivation for this thesis came primarily from MCDA applications. However, our results can be also applied in several other subfields of decision theory. In this section we discuss two of them – the state-dependent utility and the social choice problems.

4.1 Multicriteria decision analysis

MCDA provides perhaps the most natural context for our results. Indeed, in the multicriteria context the heterogeneity of the decision space dimensions is natural and the insufficiency of the previous results is apparent and has been discussed in the literature multiple times (e.g. Bouyssou et al., 2009). We have covered many aspects of the Choquet integral usage in MCDA in the previous chapters. An introduction and an example of a multicriteria model are given in Section ??, while questions of the model learning and interpretation are discussed in Section 3, together with an example of a practical application.

From the theoretical perspective, in the multicriteria context our results imply that the decision maker constructs a mapping between the elements of the criteria sets (their subsets to be precise). Some authors interpret this by saying that criteria elements sharing the same utility values present the same level of “satisfaction” for the decision maker [Grabisch and Labreuche, 2008]. Technically, such statements are meaningful, in the sense that permissible scale transformations do not render them ambiguous or incorrect, unless the representation is additive. However, the substance of the statements such as “ x_1 on criterion 1 is at least as good as x_2 on criterion 2” (which would correspond to $f_1(x_1) \geq f_2(x_2)$) is not easy to grasp. Apart from the satisfaction interpretation, perhaps one could think about workers performing various tasks within a single project. From the perspective of a project manager, achievements of various workers, serving as criteria in this example, can be level-comparable despite being physically different, if the project has global milestones (i.e. scale) which are mapped to certain personal milestones for every involved person. The novelty of our characterization is that this scale is not given a priori. Instead, we only observe preferences of the project manager and infer all corresponding mappings from them. It is also worth mentioning that value functions for any interacting pair can be seen to form a so-called Guttman scale (or a biorder) [Guttman, 1944, Doignon et al., 1984].

4.2 Psychology

An interesting connection is that in psychology there exists a body of results on the so-called cross-modality matching. A large number of studies have been conducted in this area since 1950s, with experiments related to loudness, colour, size, tone, pain, money, etc. [Stevens and Marks, 1980, 1965, Stevens, 1959, Galanter and Pliner, 1974, Krantz,

²See Lemma ?? and the definition of the interaction index given earlier in this chapter.

1972]. Kahneman [2011] gives the following example: “A girl learned to read when she was four. How tall is a man who is as tall as Julie was precocious?” Normally, kids start reading at around 5 or 6, so perhaps the girl is somewhat more precocious than average, although not by too much. Therefore, we could say that the man is somewhat higher than the average 180 cm, perhaps his height is 190 or similar. Apparently our ability to answer this question is based on the existence of some information about the distribution of the age when children start reading, and the distribution of height. The information can come in a number of forms: either just a mean value (“on average kids start reading at 5”, “an average man is 180 cm high”), or two absolute reference levels on both dimensions – “children start reading between 3 and 6”, “men heights are in the range of 165–205 cm”. Finally, we can have complete information about both distributions and pick a match based on that. It is this information that allows us to “map” four years to something like 190 cm. We can perhaps consider the probability of a certain value as the universal scale shared by two distinct elements: “75% of children start reading after 4”, “75% of men are lower than 190 cm”, etc. However, as discussed above, such information is not always available, and there might be other mechanisms by which such mappings are performed.

4.3 State-dependent utility

We will show how the traditional comonotonic-based axiomatization *implies* state-independence and how our approach can be used to construct a truly state-dependent model without making additional assumptions about correspondence between outcomes in different states.

The state-dependent utility concept, as introduced in Chapter ?? and further in Appendix ??, is evoked when the nature of the state itself is of significance and it is not assumed that outcomes in different states have the same meaning or value to the decision maker. A popular example is healthcare, where various outcomes can have major effects on the personal value of the insurance premium [Karni, 1985]. One way to model this is to use different value functions for every state; moreover, we could also consider the notion of state-prize [Karni, 1985, Karni and Schmeidler, 2016], which actually takes the state-dependent model directly to the heterogeneous product set case (dimensions are sets of “state-prizes”).

So far the axiomatizations of the state-dependent utility models have been based on the existence of some correspondence between the outcomes in different states [Karni and Schmeidler, 2016, Karni, 1993, 1985, Fishburn, 1973]. In essence, this is not different from assuming the homogeneous product set again, albeit with some technical differences (e.g. the decision space might only be a subset of the full product). Although, in principle, the existence of a preference relation on the set of state-prizes is not unrealistic, it is not clear whether this data is observable (contrary to the preferences on acts which are supposed to be always observable). Without such a relation the additive value model (think SD-EU) does not allow us to disentangle probabilities and utilities at all (see discussion in the previous section and earlier). The other question is whether this gives any real methodological advantage compared to using a union of state-prizes on every dimension and proceeding as normal. A detailed discussion of this question is given in Karni and Schmeidler [2016] and references therein, and we do not pursue it

further here. Finally, we would like to mention that the problem of state-dependence in rank-dependent models is not well developed – the only paper known to the author being Hong and Wakker [1996], where the authors comment on the meaninglessness of state-dependency in the normal CEU framework, again due to the confounding issues: “with preferences over acts as the only empirical primitive, the factorization $\nu(A)u_A(\cdot)$ becomes meaningless. Only the product $W(x, A) = \nu(A)u_A(x)$ can be derived from preferences”.

However, the general axiomatization of the Choquet integral presented in this thesis, is the first (to the author’s best knowledge) result where state-dependence can be derived *exclusively* from the preferences over acts. This constitutes a significant difference with all earlier results. As a side result, it is easy to show that comonotonicity-based conditions actually *imply* state-independence of preferences.

Lemma 6. *Let $X = Y^n$. Let conditions of the Theorem ?? hold. If for all $x \in X$ we have $i \mathbf{E}^x j$ whenever $x_i = x_j$, the representation is state-independent.*

Proof. Saying that $i \mathbf{E} j$ whenever $x_i = x_j$ in our framework amounts to saying that additive representations exist on the comonotonic subsets of X . The construction implies that $f_i(x_i) = f_j(x_j)$ whenever $x_i = x_j$. This holds for all $i, j \in X$, hence we can use a single utility function $U : Y \rightarrow \mathbb{R}$ for all dimensions. This constitutes state-independency. \square

Hence, parting with the assumption that the borders between additive regions actually coincide with the borders between *comonotonic* sets, allows us to introduce state-dependency into the model and to do so solely by observing the preferences between acts. The resulting state-dependent utility functions could be used to derive the relation on the set of state-prizes which is assumed as given in earlier works. Note that, as previously, the meaningfulness of this relation is conditional on the violation of pairwise separability in the model, as explained in Section ?. In other words, the relation might not exist between prizes of certain state pairs.

4.4 Social choice

If we think of the set N as of a society with n agents, then X is the set of all possible welfare distributions. Moreover, contrary to the classical scenario, agents could be receiving completely different goods, for example X_1 might correspond to healthcare options, whereas X_2 to various educational possibilities. In this case it is not a trivial task to build a correspondence between different options across agents. Our result basically states that provided the preferences of the social planner abide by the axioms given in Section ?, the decisions are made as if the social planner has associated cardinal utilities with the outcomes of each agent which are *unit* and *level* comparable (cardinal fully comparable or CFC in terms of Roberts [1980]). Such approach is not conventional in social choice problems, where the global (social) ordering is usually not considered as given (there are, however some papers taking this route, e.g. Ben-Porath et al., 1997). Instead, the conditions are normally given on individual utility functions and the “aggregating” functional that is used to derive the global ordering. However, one of the important questions in social choice literature is that of the interpersonal utility comparability and whether it is justifiable to assume it or not (e.g. Harsanyi, 1980). Our results show that if the

global ordering of alternatives made by the society (or the social planner) satisfy certain conditions, it is in principle possible to have individual preferences represented by utility functions that are not only unit but also level comparable with each other.

5 Summary

We have presented extensions of our characterization for the ordinal and cardinal special cases of the Choquet integral. The ordinal models are the well-known MIN/MAX and the order statistic, and also their generalization – the lattice polynomial. We have shown how these can be characterized in our framework and also related our results to the previously known axiomatizations. On the cardinal side of things, we have shown how it is possible to characterize the Choquet integral with respect to a convex capacity. The axiom is similar to the tradeoff consistency condition and is the first characterization of convex models which can deal with both cardinal and ordinal cases (or a mixture of the two).

Next, we discussed various aspects of the Choquet integral learning. Traditionally, the learning of the integral was confined to capacity learning only. However, this approach suffers from serious methodological difficulties. Namely, it requires a very strong assumption that all criteria are measured on the same scale. We looked at how various preferential information could be used in the capacity identification problem and analysed why the process of capacity identification is relatively computationally effective. In contrast, learning the capacity and the value functions together seems to be computationally very hard. There have been only a few attempts at solving it in the literature, all of them offering only some heuristic methods. Finally, we look at the problem of confounding of the value functions and the capacity. Our characterization results state that a unique decoupling of the capacity and the value functions is possible only when the dimensions of the decision space exhibit sufficient pairwise interaction. This has a profound impact on the learning properties of the Choquet integral, since it guarantees that it is impossible to obtain a unique capacity if the variables are not interacting enough, no matter how much data we have. This means that the usage of the well-known indices such as the Shapley index is limited. An alternative option is to use the “sum of Choquet” representation (??), whereby the indices become meaningful within each interaction group.

Finally, we have looked at various interpretations of our results and their applications in decision theory. We started with MCDA, which was the main inspiration for our research. Our axiomatization is a long-missing result in this area and we hope that it will help promote further theoretical research of the Choquet integral in MCDA. The characterization leads to construction of a unique mapping between elements of various criteria sets (dimensions of the decision space). This has interesting connections to the question of cross-modality mapping, which has been extensively studied in psychology since the 1950s. Finally, we discussed two other areas where our results can be applied – the social choice theory and the state-dependent DUU. The latter is especially interesting, as our characterization is the first to construct a meaningful state-dependent model based solely on the preferences among acts. Previous works introduced additional preference relations into the model, in particular the relation on the set of “state–prizes”. Conceptually, this amounts to saying that elements of various dimensions are commensurate which does not always have to be the case. Observability of this preference relation is also not apparent.

Our results do not require any additional constructs apart from the preference between acts themselves. Yet, we are able to construct a unique mapping between the outcomes in different states (provided the data exhibits sufficient interaction).

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